Q.P. 67663

MATHEMATICS SOLUTION

(DEC 2019 SEM 4 MECHANICAL)

Q1) (a) Find eigen values of $A^2 - 2A + I$ and adj A Where A = $\begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix}$. **(5M)**

Solution:

$A = \begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix}$

$A^2 - 2A + I = \begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix}\begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix} - 2\begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 16 & 5 & -7 \\ 12 & 5 & -11 \\ 24 & 8 & -12 \end{bmatrix} - \begin{bmatrix} 8 & 2 & -2 \\ 12 & 6 & -10 \\ 12 & 4 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 3 & -5 \\ 0 & 0 & -1 \\ 12 & 4 & -7 \end{bmatrix}$

The Characteristic equation is

$|A - \lambda I| = 0$

$\begin{vmatrix} 9-\lambda & 3 & -5 \\ 0 & 0-\lambda & -1 \\ 12 & 4 & -7-\lambda \end{vmatrix} = 0$

$(9-\lambda)[(-\lambda)(-7-\lambda) + 4] - 3\,[12] - 5[12\lambda] = 0$

$-\lambda(\lambda^2 - 2\lambda + 1) = 0$

$-\lambda(\lambda - 1)^2 = 0$

$\lambda = 0, 1, 1$

Hence the eigen values of $A^2 - 2A + I$ are 0, 1 and 1.

(b) A random variable X has the following probability function.

X	0	1	2	3	4
P(X=x)	1/16	4k	6k	4k	k

Find (i) k (ii) P (X < 4) (iii) P (X > 3) (iv) P (0 < X ≤ 2). **(5M)**

Solution:

(i)Since $\sum p(x_i) = 1$

1/16 + 4k + 6k + 4k + k = 1

1/16 + 15k = 1

15k = 1 – 1/16

k = 1 /16

Thus, we have the following probability distribution

X	0	1	2	3	4
P(X=x)	1/16	4/16	6/16	4/16	1/16

(ii) P (X < 4) = P (X = 0, 1, 2, 3)

= P (X = 0) + P (X = 1) + P (X = 2) + P (X = 3)

= 1/16 + 4/16 + 6/16 + 4/16

= 15/16.

(iii) P (X > 3) = P (X = 4)

= 1/16.

(iv) P (0 < X ≤ 2) = P (X = 1) + P (X = 2)

= 4/16 + 6/16

= 10/16.

(c)Can it be concluded that the average life span of an Indian is more than 71 years, if a random sample of 900 Indians has an average life span 72.8 years with standard deviation of 7.2 years? (5M)

Solution:

Null Hypothesis H_o: μ = 70 years

Alternate Hypothesis H_a: μ ≠ 70 years

Test statistic: $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

Since we are given standard deviation of the sample, we put

$\bar{X} = 71.8\,, \mu = 70, \sigma = 7.2, n = 100$

$Z = \frac{71.8-70}{7.2/\sqrt{100}} = 2.5$

Level of significance: α = 0.05

Critical value: the value of z_α at 5% level of significance is 1.96

Decision: Since the computed value $|Z| = 2.5$ is greater than the critical value z_α = 1.96, the null hypothesis is rejected.

(d) Consider the following problem

Maximize Z = $2x_1 - 2x_2 + 4x_3 - 5x_4$

Subjected to $x_1 + 4x_2 - 2x_3 + 8x_4 = 2$

$-x_1 + 2x_2 + 3x_3 + 4x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0$

Find a basic feasible solution which is non-degerate and optimal solution.

Solution:

No. of basic solutions	Non-basic variables = 0	Basic variables	Equations and values of the basic variables	Is the solution feasible?	Is the solution degenerate?	Value Of z	Is the solution optimal?
1	$x_3=0$ $x_4=0$	x_1, x_2	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0,\ x_2 = 1/2$	Yes	Yes	- 1.5	No
2	$x_2=0$ $x_4=0$	x_1, x_3 outgoing x_2 incoming x_1	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $x_1 = 8,\ x_3 = 3$	Yes	No	28	Yes
3	$x_1=0$ $x_4=0$	x_2, x_3 outgoing x_1 incoming x_2	$4x_2 - 2x_3 = 2$ $2x_2 + 3x_3 = 1$ $x_3 = 0,\ x_2 = 1/2$	Yes	Yes	-1	No
4	$x_2=0$ $x_3=0$	x_1, x_4 outgoing x_2 incoming x_1	$x_1 + 8x_4 = 2$ $-x_1 + 4x_4 = 1$ $x_1 = 0,\ x_4 = ¼$	Yes	Yes	-1.25	No
5	$x_1=0$ $x_3=0$	x_2, x_4 outgoing x_1 incoming x_2	$4x_2 + 8x_4 = 2$ $2x_2 + 4x_4 = 1$ Unbounded	---	---	---	---
6	$x_1=0$ $x_2=0$	x_3, x_4 outgoing x_2 incoming x_3	$-2x_3 + 8x_4 = 2$ $3x_3 + x_4 = 12$ $x_3 = 0,\ x_4 = 1/4$	Yes	Yes	-12.5	No

Q2(a) Check whether the given matrix A is diagonalizable, diagonalize if it is, where A = $\begin{bmatrix} 4 & 1 & -1 \\ 6 & 3 & -5 \\ 6 & 2 & -2 \end{bmatrix}$

Solution: **(6M)**

The Characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 6 & 3-\lambda & -5 \\ 6 & 2 & -2-\lambda \end{vmatrix} = 0$$

$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

$-(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$

$-(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$

λ = 1, 2, 2.

When λ = 1,

$$\begin{bmatrix} 3 & 1 & -1 \\ 6 & 2 & -5 \\ 6 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1/3$

$$\begin{bmatrix} 1 & 1/3 & -1/3 \\ 6 & 2 & -5 \\ 6 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 6R_1$

$$\begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 0 & 3 \\ 6 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 6R_1$

$$\begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2/-3$

$$\begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - (-1)R_2$

$$\begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1 - (-1/3)R_2$

$$\begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = 0$

$x_1 + \left(\frac{1}{3}\right)x_2 = 0$

Therefore,

$x_1 = \left(-\frac{1}{3}\right)x_2$

$x_2 = x_2$

$x_3 = 0$

The rank of coefficient matrix is 2. The number of unknowns is 3. Hence, there are 3-2 = 1. Putting $x_2 = t$ then $x_1 = \left(-\frac{t}{3}\right)$.

$$X_1 = \begin{bmatrix} -t/3 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$$

Corresponding to λ = 1, we get the eigenvector $X_1{}' = [-1/3 \quad 1 \quad 0]$

There are three variables and the rank is 2, hence there is only one independent solution.

For λ = 1, algebraic multiplicity is 1 and geometric multiplicity is 1.

When λ = 2,

$$\begin{bmatrix} 2 & 1 & -1 \\ 6 & 1 & -5 \\ 6 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1/2$

$$\begin{bmatrix} 1 & 1/2 & -1/2 \\ 6 & 1 & -5 \\ 6 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 6R_1$

$$\begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & -2 & -2 \\ 6 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 6R_1$

$$\begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & -2 & -2 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2/-2$

$$\begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - (-1)R_2$

$$\begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1 - (-1/2)R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_3 = 0$

$x_2 + x_3 = 0$

Therefore,
$x_1 = x_3$

$x_2 = -x_3$

$x_3 = x_3$

The rank of coefficient matrix is 2. The number of unknowns is 3. Hence, there are 3-2 = 1. Putting $x_3 = t$

$$X_1 = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

There are three variables and the rank is 2, hence there is only one independent solution.

For λ = 2, algebraic multiplicity is 2 and geometric multiplicity is 1.

Hence the algebraic multiplicity and geometric multiplicity should be same.

Thus, the matrix is not diagonalized.

(b) Verify Green's theorem for $\bar{F} = x^2 i - xyj$ where C is the triangle having vertices A (0,3), B (3,0), C (6,3). **(6M)**

Solution:

By Green's Theorem

$$\int_c P.dx + Q.dy = \iint_R \frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}.dx.dy$$

Here, P = x^2 ; Q= $-xy$

$\frac{\delta Q}{\delta x} = -y$; $\frac{\delta P}{\delta y} = 0$

(a)Along AB, since the equation of AB is

$\frac{y-3}{3-0} = \frac{x-0}{0-3}$

$y = 3 - x$

Putting P = x^2 ; Q= $-xy = -x(3-x); dy = -dx$

$$\int_c P.dx + Q.dy = \int_0^3 [x^2 + x(3-x)]dx = \int_0^3 3x.dx$$
$$= (3x^2/2|x = 0\ to\ 3)$$
$$= \frac{27}{2}$$

Along BC, since the equation of BC, $\frac{y-0}{0-3} = \frac{x-3}{3-6}$ i.e., is y = x – 3

$$\int_c P.dx + Q.dy = \int_3^6 (x^2 - x^2 + 3x)dx = \int_3^6 3x.dx$$
$$= (3x^2/2|x = 3\ to\ 6)$$
$$= \frac{81}{4}$$

Along CA, since the equation of CA is y = 3 ; dy = 0

$$\int_c P.dx + Q.dy = \int_6^0 (x^2 - x^2 + 3x)dx = \int_6^0 x^2.dx$$
$$= (x^3/3|x = 6\ to\ 0)$$
$$= -54$$

(b)

$$\iint_R \frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}.dx.dy = \int_0^3 \int_{3-y}^{3+y} (-y).dx.dy$$
$$= \int_0^3 (-y.x|y = 3 - y\ to\ 3 + y).dx$$

$$= \int_0^3 y[3 + y - 3 + y] . dx$$
$$= \int_0^1 y^3 . dx$$
$$= \left(\frac{y^4}{4} \middle| x = 0 \; to \; 3\right)$$
$$= \frac{81}{4}$$

From (a) and (b), the theorem is verified.

(c) Sample of two types of two types of electric bulbs were tested for length of life and the following data were obtained

	Type I	**Type II**
Sample size	**10**	**9**
Mean of the sample (in hours)	**1136**	**1034**
Standard deviation	**36**	**39**

Test at 5% level of significance whether the difference in the sample means is significant. (8M)

Solution:

Null Hypothesis H_o: There is no relation between the electric bulbs and length of life.

Alternate Hypothesis H_a: There is a relation between these two.

To test the significant difference between two mean $\overline{x_1}$and $\overline{x_2}$ of sample sizes n_1 and n_2 use the statistic

$$t = \frac{(\overline{x_1} - \overline{x_2})}{s\sqrt{\left(\frac{1}{n1}\right)+\left(\frac{1}{n2}\right)}}$$

$$s^2 = \frac{(n_1 s_1{}^2 + n_2 s_2{}^2)}{n_1 + n_2 - 2} = \frac{(8*36*36+7*40*40)}{8+7-2}$$

$$s = 40.7$$

$$t = \frac{(1234-1036)}{40.7*\sqrt{\frac{1}{8}+\frac{1}{7}}} = 9.39$$

Calculated t value = 9.39

Tabulated Value = 1.77 (as 5% level of significance with 13 degrees of freedom)

9.39 > 1.77, reject Ho (Null Hypothesis)

Hence, there is no relation between the electric bulbs and length of life.

Q3 (a) Use the dual simplex method to solve the following LPP

Minimise Z = $6x_1 + x_2$

Subject to $2x_1 + x_2 \geq 3$

$x_1 - x_2 \geq 0$

$x_1, x_2 \geq 0.$ **(6M)**

Solution:

Minimise $z = 6x_1 + x_2$

Subject to $-2x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq 0$

Introducing the slack variables s_1, s_2, we have

Minimise $z = 6x_1 + x_2 - 0s_1 - 0s_2$

i.e. $z - 6x_1 - x_2 + 0s_1 + 0s_2$

Subject to $-2x_1 - x_2 + s_1 + 0s_2 = -3$

$-x_1 + x_2 + 0s_1 + s_2 = 0$

Iteration Number	Basic Variable	Coeficient Of x_1	x_2	s_1	s_2	R.H.S. Solution
0	z	-6	-1	0	0	0
s_1 leaves	s_1	-2	-1*	1	0	-3
x_2 enters	s_2	-1	1	0	1	0
Ratio		3	1			
1	z	-4	0	-1	0	3
s_2 leaves	x_2	2	1	-1	0	3
x_1 leaves	s_2	-3*	0	1	1	-3
2	z	0	0	-7/3	4/3	7
	x_2	0	1	-1/3	2/3	1
	x_1	1	0	-1/3	-1/3	1

$x_1 = 1$, $x_2 = 1$ and $z_{min} = 7$

(b) Use Gauss Divergence Theorem to evaluate $\iint_S \bar{N}.\bar{F}.ds$ where $\bar{F} = 2xi + 2yj + 2z^2k$ and S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane z = 1. **(6M)**

Solution:

By divergence formula,

$\iint_S \bar{F}.d\bar{S} = \iiint_V \nabla.\bar{F}.div$

Now, $\bar{F} = 2xi + 2yj + 2z^2k$

$\nabla.\bar{F} = \left(\frac{\delta(2x)}{\delta x} + \frac{\delta(2y)}{\delta y} + \frac{\delta(2z^2)}{\delta z}\right)$
$= 2 + 2 + 2z$
$= 4 + 2z$
$= 2\,(2 + z)$

$$\iiint_V \nabla.\bar{F}.div = \iiint_V 2\,(2\ +\ z).dv = \iiint_V 2\,(2\ +\ z).dx.dy.dz$$

We shall obtain the volume integral by using cylinder co-ordinates x = r.cosθ, y = r.sinθ, z = z and dx.dy.dz = r.dr.dθ.dz

$r^2 = x^2 + y^2$ and by data $x^2 + y^2 = z^2$

z = r, Hence, z varies from 0 to 1

$$\iiint_V 2(2+z).dx.dy.dz = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} 2(2+z).r.drd\theta dz$$

$$= 2\int_0^{2\pi}\int_0^1 \left(2rz + r\frac{z^2}{2}\Big|z = r\ to\ 1\right).dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 \left(\frac{5r}{2} - 2r^2 - \frac{r^3}{2}\right).dr\,d\theta$$

$$= 2\int_0^{2\pi}\left(\frac{5r^2}{4} - \frac{2r^3}{3} + \frac{r^4}{8}\Big|x = 0\ to\ 1\right).d\theta$$

$$= 2\int_0^{2\pi}\left(\frac{5}{4} - \frac{2}{3} + \frac{1}{8}\right).d\theta$$

$$= 2\int_0^{2\pi}\frac{17}{24}.d\theta$$

$$= \left(\frac{17}{12}\theta\Big|x = 0\ to\ 2\pi\right)$$

$$= \frac{17}{12} * 2\pi = \frac{17\pi}{6}$$

(c) Find the rank, index, signature and class of the following Quadratic form by reducing it to its canonical form

$\boldsymbol{2x^2 - 2y^2 + 2z^2 - 2xy - 8yz + 6zx}$ **(8M)**

Solution:

The quadratic form can be written as

The matrix form is

$$A=\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -4 \\ 3 & -4 & 2 \end{bmatrix}$$

We write A=IAI

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -4 \\ 3 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 + \frac{1}{2}R_1,\ R_3 - \frac{3}{2}R_1,\ C_2 + \frac{1}{2}C_1,\ C_3 - \frac{3}{2}C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & -5/2 \\ 0 & -5/2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 - R_2, C_3 - C_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\frac{1}{\sqrt{2}}R_1$, $\sqrt{\frac{2}{5}}R_2$, $\frac{1}{\sqrt{2}}C_1$, $\sqrt{\frac{2}{5}}C_2$

$$\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 0 & 0\\1/\sqrt{10} & \sqrt{2/5} & 0\\-2 & -1 & 1\end{bmatrix} A \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{10} & -2\\0 & \sqrt{2/5} & -1\\0 & 0 & 1\end{bmatrix}$$

The linear transform, X = PY

$x = \frac{1}{\sqrt{2}}u + \frac{1}{\sqrt{10}}v - 2w$

$y = -\sqrt{\frac{2}{5}}u - w$

$z = w$

Transforms the given quadratic form $u^2 + v^2$

The rank = 2, Index = 1, Signature = 1

The form is positive semi-definite.

Q4 (a) Four dice were thrown 250 times and the number of appearance of 6 each time was noted

No. of successes(x):	**0**	**1**	**2**	**3**	**4**
Frequency (f):	**133**	**69**	**34**	**11**	**3**

Fit a poison distribution and find the expected frequencies for x = 1, 2, 3, 4. (6M)

Solution:

Now, mean $= m = \frac{\Sigma f_i x_i}{\Sigma f_i}$

Mean = $\frac{133*0 + 69*1 + 34*2 + 11*3 + 3*4}{250} = 0.728$

Poisson distribution of X is

P (X = x) = $= \frac{e^{-m}\ X\ m^x}{x!} = \frac{e^{-0.728}\ X\ 0.728^x}{x!}$

Expected frequency = N x p(x)

$$= 250\ X\ \frac{e^{-0728}\ X\ 0.728^x}{x!}$$

Putting x = 0, 1, 2, 3, 4 we get the expected frequencies as 133, 69, 34, 11, 3.

f (x + 1) = $\frac{m}{x+1}\ .f(x) = \frac{0.728}{x+1}\ .f(x)$

Putting x = 0, f (1) = $\frac{0.728}{0+1} * 69 = 50$

Putting x = 1, f (2) = $\frac{0.728}{1+1} * 34 = 12$

Putting x = 2, f (3) = $\frac{0.728}{2+1} * 11 = 3$

Putting x = 3, f (4) = $\frac{0.728}{3+1} * 3 = 1$

(b) Verify Cayley Hamilton theorem for matrix A and hence find the matrix represented by $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 11I$

Where A = $\begin{bmatrix} 3 & -2 & 3 \\ 10 & -3 & 5 \\ 5 & -4 & 7 \end{bmatrix}$. **(6M)**

Solution:

The characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -2 & 3 \\ 10 & -3-\lambda & 5 \\ 5 & -4 & 7-\lambda \end{vmatrix} = 0$$

$-\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$

By Cayley Hamilton theorem, this equation is satisfied by A

$-A^3 + 7A^2 - 16A + 12I = 0$

$$A^2 = \begin{bmatrix} 3 & -2 & 3 \\ 10 & -3 & 5 \\ 5 & -4 & 7 \end{bmatrix}\begin{bmatrix} 3 & -2 & 3 \\ 10 & -3 & 5 \\ 5 & -4 & 7 \end{bmatrix} = \begin{bmatrix} 4 & -12 & 20 \\ 25 & -31 & 50 \\ 10 & -26 & 44 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4 & -12 & 20 \\ 25 & -31 & 50 \\ 10 & -26 & 44 \end{bmatrix}\begin{bmatrix} 3 & -2 & 3 \\ 10 & -3 & 5 \\ 5 & -4 & 7 \end{bmatrix} = \begin{bmatrix} -8 & -52 & 92 \\ 15 & -157 & 270 \\ -10 & -118 & 208 \end{bmatrix}$$

$$-A^3 + 7A^2 - 16A + 12I = -\begin{bmatrix} -8 & -52 & 92 \\ 15 & -157 & 270 \\ -10 & -118 & 208 \end{bmatrix} + 7\begin{bmatrix} 4 & -12 & 20 \\ 25 & -31 & 50 \\ 10 & -26 & 44 \end{bmatrix} - 16\begin{bmatrix} 3 & -2 & 3 \\ 10 & -3 & 5 \\ 5 & -4 & 7 \end{bmatrix} + 12\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) An investigation into the equality of standard deviation of two normal populations gave the following results.

Sample	Size	Sample mean	Sum of squares of deviations from the mean
1	13	18	105
2	21	24	145

Examine the equality of sample variances at 5% level of significance.
(Given: $F_{0.025}$ = 2.68 for d.o.f. 12 and 20 and $F_{0.025}$ = 3.07 for d.o.f. 20 and 12) **(8M)**

Solution:

Null Hypothesis Ho: $\sigma_1^2 = \sigma_1^2$
Alternative Hypothesis Ha: $\sigma_1^2 \neq \sigma_1^2$

Calculations of Test Statistic: $F = \frac{n_1 s_1^2/(n_1-1)}{n_2 s_2^2/(n_2-1)}$

But $n_1 {s_1}^2 = \sum(x_i - \bar{x})^2$ and $n_2 {s_2}^2 = \sum(y_i - \bar{y})^2$

F=$\frac{105/12}{145/20} = \frac{8.75}{7.25} = 1.207$
Level of significance α=0.05

Degree of freedom $v_1 = n_1 - 1 = 12$ for the numerator
$v_2 = n_2 - 1 = 20$ for the denominator
Critical Value: The table value
$F_{(12,20)}(0.025)=2.68$
$F_{(20,12)}(0.025)=3.07$
$\frac{1}{F(20,12)(0.025)} = \frac{1}{3.07} = 0.326$
Decision: Since the calculated value F=1.207 lies between 0.326 and 3.07, we accept the null hypothesis.

Q5 (a) Is matrix A = $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & -1 \\ 3 & -1 & 3 \end{bmatrix}$ Derogatory matrix? Find its minimal polynomial. (6M)

Solution:

The Characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -3 & 3-\lambda & -1 \\ 3 & -1 & 3-\lambda \end{vmatrix} = 0$$

$(2-\lambda)[(3-\lambda)(3-\lambda) - 1] = 0$

$-(2-\lambda)(\lambda^2 - 6\lambda + 8) = 0$

$-(2-\lambda)(2-\lambda)(4-\lambda) = 0$

λ = 2, 2, 4.

Let us find the minimal polynomial of Awe know that each characteristic root of A is also a root of the minimal polynomial of A. So if f(x) is the minimal polynomial of A, then x – 2 and x – 4 are the factors of f(x). Let us see whether (x - 2)(x - 4) = x^2 - 6x + 8 annihilates of A.

Now, $A^2 - 6A + 8I$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & -1 \\ 3 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & -1 \\ 3 & -1 & 3 \end{bmatrix} - 6\begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & -1 \\ 3 & -1 & 3 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -18 & 18 \\ 0 & 10 & -6 \\ 0 & -6 & 10 \end{bmatrix} - 6\begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & -1 \\ 3 & -1 & 3 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

f(x) = x^2 - 6x + 8 annihilates A.

Thus, f(x) is the monic polynomial of the lowest degree that annihilates A.

Hence, f(x) is the minimal polynomial of A. Since the degree of f(x) is less than the order of A. A is derogatory.

(b) A vector field $\bar{F}$ is given by

$$\bar{F} = (y\sin x - \sin x)i + (x\sin z + 2yz)j + (xy\cos z + y^2)k$$

Prove that $\bar{F}$ is irrotational. Hence find its scalar potential function ϕ and ϕ (π,1,0). **(6M)**

Solution:

We have,

$$\text{Curl } \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ ysinx - sinx & xsinz + 2yz & xycosz + y^2 \end{vmatrix}$$

$$= i\left(\frac{\delta}{\delta y}(xycosz + y^2) - \frac{\delta}{\delta z}(xsinz + 2yz)\right) - j\left(\frac{\delta}{\delta x}(xycosz + y^2) - \frac{\delta}{\delta z}(ysinx - sinx)\right)$$
$$+ k\left(\frac{\delta}{\delta x}(xsinz + 2yz) - \frac{\delta}{\delta y}(ysinx - sinx)\right)$$

$$= [xcosz + 2y - xcosz - 2y]i + [ycosz - ycosz]j + [sinz - sinx]k$$

$$= 0i + 0j + 0k$$

Hence, $\bar{F}$ is irrotational.

If ϕ is the scalar potential then $\bar{F} = \nabla\phi$

$$(y\sin x - \sin x)i + (x\sin z + 2yz)j + (xy\cos z + y^2)k = \frac{\delta\phi}{\delta x}i + \frac{\delta\phi}{\delta y}j + \frac{\delta\phi}{\delta z}k$$

$\frac{\delta\phi}{\delta x} = (y\sin x - \sin x)$……………………………..(i)

$\frac{\delta\phi}{\delta y} = (x\sin z + 2yz)$……………………………..(ii)

$\frac{\delta\phi}{\delta z} = (xy\cos z + y^2)$……………………………..(iii)

But, dϕ = $\frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz$

$$= (y\sin x - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$$

$$= [ysinz\ dx + xsinz\ dy + xycosz\ dz] + (-sinx)dx + (2yz\ dy + y^2dz)$$

By integration, $\phi = xysinz + cosx + y^2z + c$ where c is the constant of integration.

Hence putting x = π, y = 1 and z = 0

Now, ϕ (π,1,0) = $\pi * 1 * sin0 + cos\pi + (1)^2 * 0 + c$

$$= cos\pi + c$$

(c) The following table gives the result of opinion pole for three vehicles A, B, C. Test whether the age and the choice of the vehicle are independent at 5% level of significance using chi – test.

Age	**Vehicle**			**Total**
	A	**B**	**C**	
20-35	**25**	**40**	**35**	**100**
35-50	**35**	**24**	**41**	**100**
Above 50	**40**	**36**	**24**	**100**
Total	**100**	**100**	**100**	**300**

Solution: **(8M)**

Null Hypothesis H_o: There is no relation between the age and choice of vehicle.

Alternate Hypothesis H_a: There is a relation between these two.

On the basis on this hypothesis the number in the first cell $= \frac{A\ X\ B}{N}$

where, A = total in the first column

B = total in the first row,

N = total number of observations

The number in the first cell of the first row $= \frac{100\ X\ 100}{300} = 33.33$

Similarly, The number in the second cell of the first row $= \frac{100\ X\ 100}{300} = 33.33$

The number in the first cell of the second row $= \frac{100\ X\ 100}{300} = 33.33$

Since the total remain the same, the numbers in the remaining cells are 33.33

Thus, we get the following table

Age	Vehicle			Total
	A	B	C	
20-35	33.33	33.33	33.33	100
35-50	33.33	33.33	33.33	100
Above 50	33.33	33.33	33.33	100
Total	100	100	100	300

Calculation of $(O - E)^2/E$

O	E	O - E	$(O - E)^2$	$(O - E)^2/E$
25	33.33	-8.33	69.39	2.082
35	33.33	1.67	2.79	0.083
40	33.33	6.67	44.49	1.335
40	33.33	6.67	44.49	1.335
24	33.33	-9.33	87.05	2.612
36	33.33	3.33	11.09	0.333
35	33.33	2.33	5.43	0.163
41	33.33	7.67	58.43	1.753
24	33.33	-9.33	87.05	2.612
			Total	$x^2 = 12.308$

Level of significance: $\alpha = 0.05$

Degree of freedom: $(r - 1)(c - 1) = (3 - 1)(3 - 1) = 4$

Critical Value: For 4 degree of freedom at 5% level of significance, the table value of $x^2 = 9.488$

Decision: Since the calculated value $x^2 = 12.308$ is more than the table value $x^2 = 9.488$, the hypothesis is not accepted.

There is a relation between the age and the choice of the vehicle.

Q6 (a) State stokes theorem and evaluate $\int [(x^2 + y^2)i + (x^2 - y^2)j].d\bar{r}$ where C is the square in the xy – plane with vertices (1,0), (0,1), (-1,0) and (0,-1). (6M)

Solution:

By Stokes theorem $\int_c \bar{F}d\bar{r} = \iint_s \bar{N}.\nabla.\bar{F}ds$

Now, $\nabla\, X\, \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ (x^2+y^2) & (x^2-y^2) & 0 \end{vmatrix} = (0-0)i - (0-0)j + (2x-2y)k$

$= (2x - 2y)k$

$\bar{N}.\nabla.\bar{F}ds = (2x - 2y)dx.dy$

$\iint_c \bar{N}.\nabla\, X\, \bar{F}.\, ds = \iint_c (2x - 2y).dx.dy$ where C is the square ABCD

$= 4\iint_{\Delta OAB}(2x - 2y).dx.dy$

The equation of the line AB is $\frac{y-1}{1-0} = \frac{x-0}{0-1}$ i.e. $y = 1 - x$

$\int_0^1 \int_{y=0}^{1-x}(2x - 2y).dy.dx = \int_0^1 \left(2xy - 2 * \frac{y^2}{2}\Big| y = 0\ to\ 1 - x\right)$.dx

$= \int_0^1 2x(1-x) - 2 * \frac{(1-x)^2}{2}$.dx

$= \int_0^1 2x - 2x^2 - 2 * \frac{(1-2x+x^2)}{2}.dx$

$= \int_0^1 -3x^2 - 1$.dx

$= \left(-\frac{3x^3}{3} - x\Big| x = 0\ to\ 1\right)$

$= -2$

(b) Monthly salary X is an organisation is normally distributed with mean Rs. 3000 and standard deviation of Rs 250. What should be the normally minimum salary of an employee in this organisation so that the probability that an employee to top 5% employees? (6M)

Solution:

Mean(m) = 3000

Standard deviation (σ) = 250
Let X denote monthly salary of a worker.

Let X_1 be the minimum salary of the top 5% workers.
Let z_1 be the corresponding pnv.

$P(X > X_1) = 5$

$P(Z > Z_1) = 0.05$

∴Area between z = 0 to z = 5
= 0.5 - 0.05 = 0.45
From table

$z_1 = 1.645$

$\frac{X_1 - m}{\sigma} = 1.645$

$\frac{X_1 - 3000}{250} = 1.645$

$X_1 = 3411.25$

Hence the minimum salary of the top 5% workers = Rs 3411.25

(c) Using duality solve the following LPP

Maximize $Z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 5$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0.$ **(8M)**

Solution:

The dual of the given problem is

Minimize $w = 5y_1 + 3y_2$

Subject to $2y_1 + y_2 \geq 5$

$y_1 + y_2 \geq 3$

$y_1, y_2 \geq 0.$

Introducing the slack and artificial variables, the problem becomes

Minimize $w' = -w = -5y_1 - 3y_2$

i.e. $w' = -5y_1 - 3y_2 - 0s_1 - 0s_2 - MA_1 - MA_2$………………(i)

subject to $2y_1 + y_2 - s_1 - 0s_2 + A_1 - 0A_2 = 5$…………………………(ii)

$y_1 + y_2 - 0s_1 - s_2 - 0A_1 + A_2 = 3$…………………………..(iii)

Multiply (ii) and (iii) by M and add to (i)

$w' = -5y_1 - 3y_2 + 3My_1 - 2My_2 - Ms_1 - Ms_2 - 0A_1 - 0A_2 - 5M$

$w' + (5 - 3M)y_1 + (3 - 2M)y_2 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = -5M$

Iteration Number	Basic Var.	Coeff icient of						R.H.S. solution	Ratio0
		y_1	y_2	s_1	s_2	A_1	A_2		
0	w’	5-3M	3-2M	M	M	0	0	-5M	
A_1leaves	A_1	2*	1	-1	0	1	0	3	3/2
y_1 enter	A_2	1	1	0	-1	0	1	2	2
1	w’	0	(1-M)/2	(5-M)/2	M	(3M-5)/2	0	-(15+M/)2	
A_2leaves	y_1	1	1/2	-1/2	0	1/2	0	3/2	3
y_2 enter	A_2	0	½*	1/2	-1	-1/2	1	1/2	1
2	w’	0	0	2	1	M-2	M-1	-8	
	y_1	1	0	-1	1	1	-1	1	
	y_2	0	1	1	-2	-1	1	1	

Since, $s_1 = 2$, $s_2 = 1$ and $w'_{max} = -8$, therefore $w'_{min} = 8$

$x_1 = 2$, $x_2 = 1$ and $z_{max} = -8$.